



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

for $\frac{d\zeta}{dt}$, which is therefore equal to the constant term of $-\frac{n'^2}{n} \frac{a'^3}{r'^3}$, that is, to

$$-\frac{n'^2}{n} \left(1 + \frac{3}{2} e'^2\right). \quad \text{Thus}$$

$$\frac{d\zeta}{dt} = n - \frac{n'^2}{n} \left(1 + \frac{3}{2} e'^2\right) - \left(\frac{3}{2} \frac{n'^2}{n} - \frac{3675}{64} \frac{n'^4}{n^3}\right) \delta \cdot e'^2.$$

We could have added to the first two terms of this equation a term $B \frac{n'^4}{n^3} e'^2$,

where B is a numerical coefficient, equal to the aggregate of the constants we have virtually neglected whenever we wrote $\delta \cdot e'^2$ for e'^2 , but it will be easily seen that this would not change the final result. We evidently have

$$n_0 = n - \frac{n'^2}{n} \left(1 + \frac{3}{2} e'^2\right).$$

From which, to a sufficient degree of approximation,

$$n = n_0 + \frac{n'^2}{n_0}.$$

Substituting this value of n , we get

$$\frac{d\zeta}{dt} = n_0 \left[1 + \left(\frac{3}{2} \frac{n'^2}{n_0^2} - \frac{3771}{64} \frac{n'^4}{n_0^4} \right) (e'^2_0 - e'^2) \right].$$

DISCUSSION OF THE GENERAL EQUATION OF THE THIRD DEGREE.

BY JOHN BORDEN, CHICAGO, ILL.

$$x^3 + Ax^2 + Bx + C = y, \tag{1}$$

$$3x^2 + 2Ax + B = \frac{dy}{dx}, \tag{2}$$

$$6x + 2A = \frac{d^2y}{dx^2}. \tag{3}$$

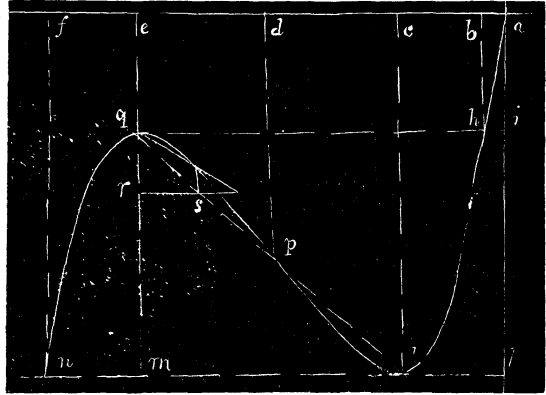
$$\text{If } \frac{dy}{dx} = 0, \text{ then, in (2),} \quad x = -\frac{A}{3} \pm \sqrt{\left(-\frac{B}{3} + \frac{A^2}{9}\right)}; \tag{4}$$

$$\text{and if } \frac{d^2y}{dx^2} = 0, \text{ then, in (3),} \quad x = -\frac{A}{3}. \tag{5}$$

First Case; A and B positive and $B < \frac{1}{3}A^2$.—The form of the curve is as in the Fig.; its locus being taken for $C = 0$.

1. The curve has an inflexion at p ; $ad = -\frac{1}{3}A$. And from p to the right the curve is concave to the line cl , from p to the left it is concave to the line qm . Revolve the left hand branch on the point p and it will coincide with the right hand branch, q falling on l .

2. The form of the curve is not dependent on the term C . Move the curve up and down so that the points q and l follow the lines em & cl and C will vary. For C positive all the roots are negative; and for C negative, one of the roots is positive. When q coincides with e a case of equal roots occurs, the third root being



ab . When l coincides with c another case of equal roots occurs, the third root being af . When cl and eq are both negative the only real root refers to the curve from h upward. When cl and eq are both positive the only real root refers to the curve from n downward.

3. The values of x , in (4), are ac and ae . Substitute these values in (1) and cl and eq are found for any assumed value of C . If the results give the same sign there is only one real root, if different signs, there are three real roots. If the values of x in (4) are real, and when substituted in (1) give contrary signs, there are three real roots, otherwise only one real root.

4. To find ab , substitute ae in (1) and make C such as to reduce y to zero; the third root is ab , so, to find af , substitute ac in (1) and make C such as to reduce y to zero, and the third root is af . Or deduct twice ae from A and the result is ab , so also, deduct twice ac from A and the result is af .

5. When all the roots are real, one lies between ab and ac , one between ac and ae and one between ae and af .

To find the middle root, suppose C in (1) to have such value that rs will coincide with the axis of x , then $ae - rs$ would be the first approximation, and $rs = (ml \times qr) \div qm$, all of which are known, for qr is what eq becomes in such case. The line rs , if above the point of inflexion, is too little as the figure shows. The tangent to the curve at the point where the ordinate at s cuts the curve makes an angle with rs whose tangent is found by substituting $ae - rs$ for x in (2). The vertical line of the triangle is the value of y in (1) when $ae - rs$ is substituted for x in (1). Hence, by proportion

we reach a formula for the base of the triangle which is the same as Horner's rule. This, if added to rs , evidently makes it too much when rs is above the point of inflexion. An approximation for the real root between ac and ab , and for the one between ae and af can be found by proportion. The curve is distorted in the figure to show the curvature, it in fact approaches more nearly to a straight line.

If C is positive and cl and eq are both negative, the only real root lies between zero and ab , which can be found in the same manner. If C is negative and cl and eq are both negative, the only real root lies between zero and $-C \div B$. If C is positive and cl and eq are both positive, then the only real root is greater than af .

To find the other limit of this root, substitute af for x in (2) and we thus find the tangent of the angle which the tangent to the curve at the point n makes with the axis of abscissas. The curve in such cases cuts the axis of x to the right of the point where this tangent cuts it; and by proportion we find, when the tangent cuts that axis, the vertical of the triangle whose base we seek $= C - qm$.

Secod Case; A negative, B positiv, and $B < \frac{1}{3}A^2$.—In this case substitute $-x$ for x , and we have the previous case.

Third Case; B negative.—In this case the axis of y lies between cl and eq .

Fourth Case; B positive and $B > \frac{1}{3}A^2$.—In this case the values of x in (2) are imaginary. The points q and l have met at the point of inflexion and the curve from p to the right is concave to the line dp , and from p to the left it is concave to the extension of that line below the point p . There can be only one real root, whose locus can be fixed in the manner above indicated.

The higher equations can be treated in the same way. Take the biquadratic

$$x^4 + Ax^3 + Bx^2 + Cx + D = y, \quad (6)$$

$$4x^3 + 3Ax^2 + 2Bx + C = \frac{dy}{dx}, \quad (7)$$

$$12x^2 + 6Ax + 2B = \frac{d^2y}{dx^2}. \quad (8)$$

If (7), for $\frac{dy}{dx} = 0$, has only one real root, then (6) cannot have more than two real roots. But if $\frac{dy}{dx} = 0$ gives three real values for x in (7) which when substituted in (6) gives consecutively contrary signs, then equation (6) has four roots for $y = 0$.